THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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NOTICE.

The following Reports have been issued by the Association:—(i) "Revised Report on the Teaching of Elementary Algebra and Numerical Trigonometry" (1911), price 3d. net; (ii) "Report on the Correlation of Mathematical and Science Teaching," by a Joint Committee of the Mathematical Association and the Association of Public School Science Masters, price 6d. net; (iii) A General Mathematical Syllabus for Non-Specialists in Public Schools, price 2d. net.; (iv) Report on the Teaching of Mathematics in Preparatory Schools, 1907, price 3d. net. These reports may be obtained from the Publishers of the *Gazette*.

(v) **Catalogue of current Mathematical Journals**, with the names of the Libraries where they may be found. Pp. 40. Price, 2s. 6d. net.

"Even a superficial study convinces the reader of the general completeness of the catalogue, and of the marvellous care and labour which have gone to its compilation."—*Science Progress*, Jan. 1916.

(vi) **Report of the Girls' Schools Committee, 1916**: Elementary Mathematics in Girls' Schools. Pp. 26. 1s. net.

(vii) **Report on the Teaching of Mechanics, 1918** (*Mathematical Gazette*, No. 137). 1s. 6d. net.

(viii) **Report on the Teaching of Mathematics in Public and Secondary Schools, 1919** (*Mathematical Gazette*, No. 143). 2s. net.

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A SCHOOL COURSE IN SURVEYING.

BY H. R. VERNON, M.A.

THE Course described here has been evolved at Northallerton Grammar School during the past nine years. The field-work and plan-drawing are done during Geography lessons, while the calculation of area, height finding and resection are undertaken by the mathematics classes. Many mathematical ideas are involved throughout the Course, which acts as a connecting link between the two subjects. The ages given are only approximate.

PART I.

Involving straight lines and right-angles only.

Age 10-11.

(i) *Plan of Class Room.* Copies of simple rectilinear figures to a scale of $\frac{1}{4}$, $\frac{1}{2}$, and other simple fractions have been drawn in early mathematics lessons. The length and breadth of the room are measured in lengths of foot of a member of the class (to the nearest whole number), and an approximate drawing made on the board, as in Fig. 1, which brings out the fact that measurements are wanted other than those of length. Since the only angles involved

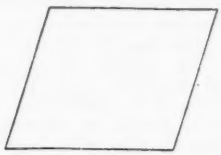

FIG. 1.

here, however, are right angles that point is passed over for the present, but the pupils are made to realise that the 'shape'—the term 'similarity' is not introduced till later—is the important fact of the plan. The golden rule for choosing scale is that it shall be as *conveniently* large as the paper will allow. In this case it is found that the paper will take about half a foot-length, so that if the room measures 22 foot-lengths the scale chosen would be 1/40. A line is then drawn half a foot-length long (representing 20 foot-lengths) and divided up into convenient smaller lengths, say, representing 5 foot-lengths. The plan is then drawn by means of this scale, and the length of the diagonal found from the plan and checked by measuring on the floor. By placing

several of the plans end to end on the floor it is easily seen that 40 would be required to make a complete row across the room, while to cover the floor 40 such rows or 1600 plans would be necessary, so that the areal scale is not 1/40 but $(1/40)^2$. Other details, such as door, windows, etc., are inserted. Various lengths on the plan are then measured in inches and in centimetres, and it is found that by multiplying these lengths by 40 the real lengths can be found *whatever units are used*. This fact being realised, a scale is drawn representing feet, and an approximate relation in the form of feet to the inch is calculated.

(ii) *Plan of Tennis Court.* This is made first by pacing and later by measuring with a 100-foot tape.

(iii) *Plan of the School Buildings.* These consist of two blocks involving only right angles, but a new difficulty is introduced in getting the two blocks placed correctly relative to one another. Plan A (Fig. 2) is drawn first. Then points x and y are fixed by measuring mw, wx and nz, zy, thus introducing in a simple manner the method of offset surveying which is used in Part II. The line xy being established, the plan of B is easily completed.

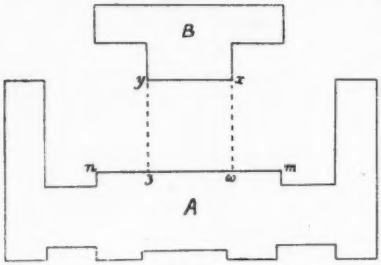

FIG. 2.

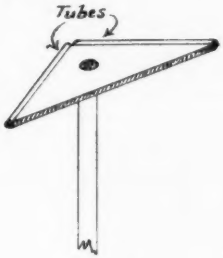

FIG. 3.

PART II.

Age 11-12.

(i) *Plan of the School Grounds.* This is accomplished by an offset survey, which has been found to be the most easily understood and the most accurate method and requiring only the simplest instruments. The grounds are flat, so that fortunately no difficulties arise from difference of level. The chief instrument used is a large right-angle-45° set square about $8'' \times 8''$ fixed on a broom handle about 4′ long. Two wooden thermometer cases with the bottoms cut off are fastened along the perpendicular sides and act as sights (Fig. 3). A main line AB (Fig. 4) is chosen along the front of the school as being long, central and easily picked up anywhere in the grounds. The instrument is held by a (Fig. 4), who sights it along AB, while b, looking

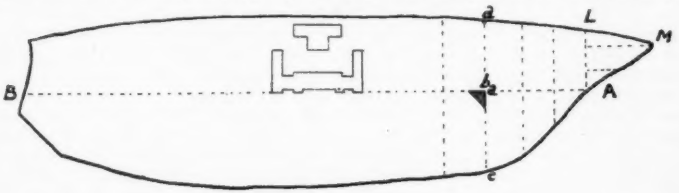

FIG. 4

through the other sight, moves c into position on the boundary. c then moves d into line with b. Lengths are then measured with the 100-foot tape along and perpendicular to the main line. Results are entered into a field book in the usual manner (Fig. 5). Corner ALM is surveyed by using AL

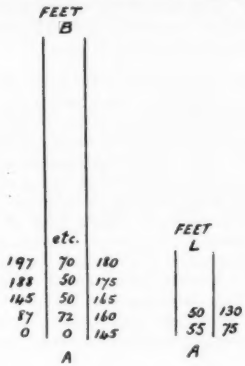

FIG. 5.

as a main line and taking offsets from it. The figures in the centre column are added up to give length AB, and the greatest width is found by adding corresponding offsets. This enables the scale to be chosen to fit the paper. The scale is exhibited in its three forms: R.F. 1/600: 50 feet to 1 inch:

(ii) Some height finding is done during this year, angles being measured by simple clinometer (Fig. 6), and heights found by drawing to scale.

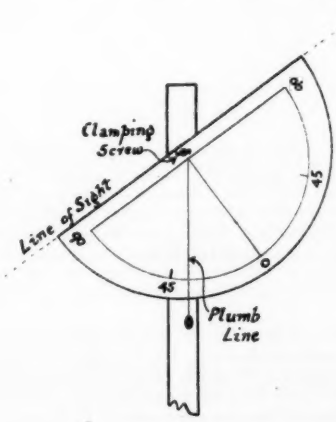

FIG 6

PART III.

Age 12-13.

(i) *Calculating Areas.* The area of the school grounds is calculated from the figures obtained in the offset survey of Part II., regarding the area as being split up into a series of trapezia and triangles.

(ii) *Plane-tabling.* Up to the present the difficulty of measuring and reproducing angles has been avoided by using only right angles. The plane-table is now introduced as a very simple method of overcoming the difficulty. The grounds are not systematically surveyed by this method, as it has been found to be less accurate (with the rough instruments at our disposal) than the offset method, though it was used during the war to survey a part of the grounds under crops, and which therefore could not be traversed. The table is now used to plot the position of points from a distance and to illustrate the proportionality of similar triangles. A brief description of the method of working may be of interest. Instruments required are a flat table which can be easily transported, levelled and rotated—a drawing-board fastened by a thumb-screw to a strong camera-stand serves the purpose excellently—a ruler with a sighting arrangement and a circular or T-spirit level. Select a long level base line AB (Fig. 7), mark the ends by posts and carefully measure

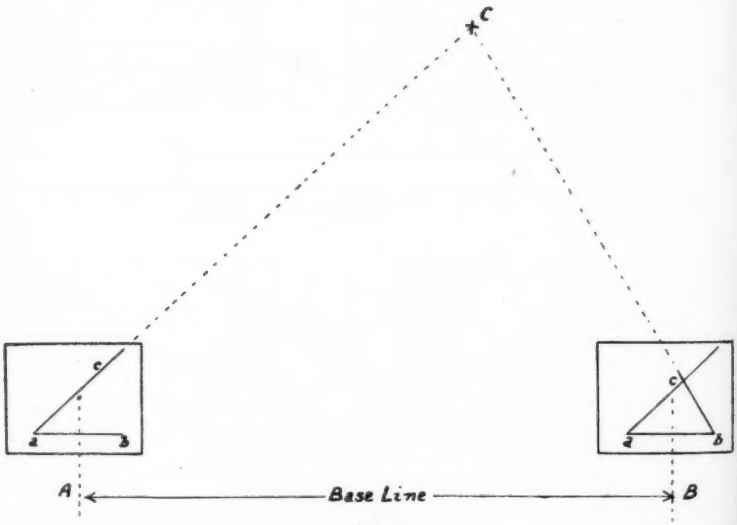

FIG. 7.

it. Pin a sheet of paper to the table, and draw a line ab to represent it to a suitable scale. Set up the table at A, level it, lay the sight-ruler along ab and rotate the table until the direction of ab corresponds with that of AB. Clamp the table, lay the rule through a and sight it towards a distant point C: draw a ray ac. Repeat for any other points D, E, etc., whose positions are required. Set up the table at B, level and rotate until ba corresponds

with BA ; * then draw a ray through b towards C cutting ac in c, which gives the position of C ; and so for points D, E, etc. Any of the lines AC, BD, etc., can now be used as base lines. By this method a plan can be constructed in the field.

(iii) Heights are also calculated during the year by trigonometry, angles being measured by clinometer or sextant.

PART IV.

Resecting Observer's Position. This is carried out on the 6″ and 1″ Ord. Survey Maps by two methods which do not require a knowledge of the amount of the magnetic variation. The class works from a spot not clearly defined on the map, *e.g.* the middle of a large field.

(*a*) *Two-circle Method.*† Three distinct points ABC (Fig. 8) marked on the map are identified, and the angles AOB ($=\alpha$ say), BOC ($=\beta$ say) are measured

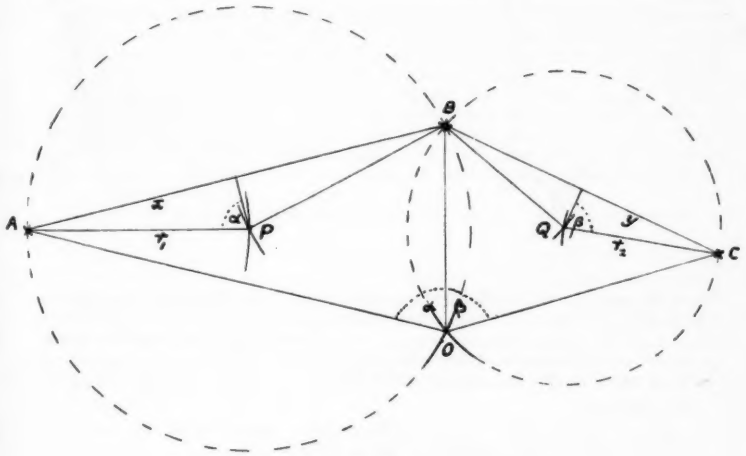

FIG. 8.

(by prismatic compass or sextant), where O is the observer's position. On the map measure AB ($=2x$ say), BC ($=2y$ say). From Fig. 8 it will be seen that $r_1 = x/\sin . \alpha$, $r_2 = y/\sin . \beta$. On the map draw arcs radius r_1 with centres A and B intersecting in P, and arcs radius r_2 with centres B and C intersecting in Q. With centre P and radius r_1, draw a circle cutting the circle centre Q and radius r_2 in O, which is the required point. In practice the position of O is approximately known, and there is seldom any doubt as to which side of AB, BC, P and Q lie. This method is accurate and provides some interesting mathematics.

---

* The table can also be oriented by a compass. A magnetic meridian is drawn when the table is in position at A. At any other position it is oriented by rotating until the compass needle corresponds with this meridian. This is perhaps a more accurate method for extended surveys.

† This method was described by Capt. Morrison-Scott, R.M., in the *Journal of the Royal Artillery*, Aug. 1917.

(*b*) *Triangle of Error Method.** The map is pinned to the plane-table and the estimated position of the observer is marked (*o*, Fig. 9). Three known points *A*, *B*, *C* marked (*a*, *b*, *c*) on the map are identified. The sight-rule is laid along *oa*, a line drawn and the table rotated until the rule is pointing to *A*. In general this orientation is incorrect. The table is clamped and the rule placed on *b* and sighted to *B*, and a ray drawn backwards from *b* towards *o*: similarly for *C*. If the orientation is correct, the three rays will be concurrent, though not necessarily at *o*, but if not, a small triangle will be formed. From this triangle the true position *p* can be deduced. There are two cases—when *p* lies (i) within, (ii) without, the triangle *abc*. Case (i), *p* lies within the triangle of error; case (ii), *p* lies without the triangle of error in one of the six sectors formed by the rays, such that *p* is on the same side of the rays looking towards *a*, *b* and *c*. In both cases it is at a distance from any ray proportional to its distance from the corresponding point. It will be seen from Fig. 9 that the error ϵ of orientation has the effect of deflecting

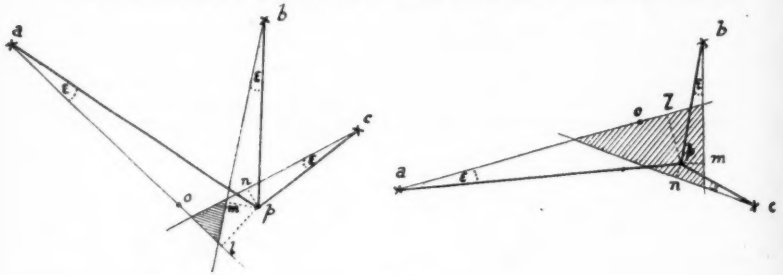

FIG. 9.

the true rays *ap*, *bp*, *cp* through the same angle ϵ, in the same direction, to the positions *al*, *bm*, *cn*. Hence the triangles *apl*, *bpm*, *cpn* are similar, and *pl*, *pm*, *pn* are proportional to *pa*, *pb*, *pc*. The process is repeated for point *p*, and so on until absolute concurrency is obtained. This is a quick, practical method which can be carried out in the field.† H. R. VERNON.

Northallerton.

GLEANINGS FAR AND NEAR.

85. The Senior Moderator in the Tripos of 1856 was (Sir) John Budd Phear. An invalid who had spent much valuable time on one of the problems (? Bishop Forrest Browne) perpetrated the epigram:

May Phear ne'er metus face to face
At any other time or place.

The other Moderator, William Henry Besant, brother of the novelist, set a question which involved an explanation of the action of the inverted cup in a tart. One man got full marks. Another treated it as a matter of barometric pressure, and happening to know that the water barometer has a column of about 34 feet, did not forget, as so many less cautious candidates might have done, to add that owing to this fact the cup in the tart need not be more than 34 feet high.—Bishop Browne, *Recollections*. 1915.

* During the late war the Royal Engineers used this method for resecting battery positions and other work requiring considerable accuracy.

† There is a case of failure by both methods when *A*, *B*, *C*, *O* or *a*, *b*, *c*, *p* are concyclic.

MISSING-FIGURE PROBLEMS.

By W. E. H. Berwick.

In a recent number of the *Mathematical Gazette* (March 1920) I proposed the following problem for solution:

THE FOUR FOURS.

$$
\begin{array}{rl}
\times\times\times)\times\times\times\times\times\times\times 4(\times 4\times\times & (1)\\
\quad\times\times\times & (2)\\
\quad\overline{\times\times 4\times} & (3)\\
\quad\times\times\times\times & (4)\\
\quad\quad\times\times\times\times & (5)\\
\quad\quad\times 4\times & (6)\\
\quad\quad\overline{\times\times\times\times} & (7)\\
\quad\quad\times\times\times\times & (8)
\end{array}
$$

Some of the figures in the long division sum have been replaced by crosses, and it is required to restore them. Four solutions are possible—of course in scale ten. It is understood that the sum is to be worked as in elementary arithmetic and that there is nothing fictitious. There must be no line of zeros, neither must the left-hand figure of any line be zero. In the divisor the digits figure ought not to be zero (for such a zero would ordinarily be cancelled before working the sum), but it is not enough to assume this: proof must be possible from the data. In the FOUR FOURS the proof is instantaneous, because there is no remainder and the right-hand figure of the dividend is 4.

The reasoning which gives rise to the solution of the FOUR FOURS is as follows:

(i) Let the divisor be $D \equiv abc$ and the quotient $Q \equiv d4ef$, the notation being that of positive integers in scale ten. Since $e \,.\, D$ and $4 \,.\, D$ are respectively three-figure and four-figure numbers [from lines (6) and (4)], e cannot exceed 3. Further $(e+1) \,.\, D \geqslant 10^3$ otherwise, from line (5), the third figure in the quotient would be $e+1$ at least.

(ii) Trying $e=3$, $3 \,.\, D = \times 4 \times$ and $4 \,.\, D \geqslant 1000$. The sixth line is $84\times$ or $94\times$ and $f \geqslant 4$. Since $cf \equiv 4 \pmod{10}$ from the 4 in the dividend we have as possibilities for D and Q:

D	- 280,	281,	282,	283,	314,	315,	316,	316,
Q	- —	$\times 434$,	$\times 437$,	$\times 438$,	$\times 436$,	—	$\times 434$,	$\times 439$.

All these have to be rejected through their failure to satisfy one (or both) of the considerations:

(α) $D \,.\, ef$ is a five-figure number from line (5),
(β) the third figure in $D \,.\, 4ef$ is 4 from line (3).

Accordingly $e \neq 3$.

(iii) Next trying $e=2$, reasoning similar to that in (ii) gives the following possibilities for D and Q:

D	-	370,	371,	372,	373,	374,
		420,	421,	422,	423,	424,
		470,	471,	472,	473,	474,
Q	-	—	$\times 424$,	$\times 427$,	$\times 428$,	$\times 426$.

All fail to satisfy one or both of (α), (β) in (ii). Hence $e \neq 2$, and the only remaining choice is $e=1$.

(iv) When $e=1$, $a > 4$ from line (5), $b=4$, $d=1$ and $Q < 1500$. If $a=5$ or 6, $D = 54\times$ or $64\times$ and $D \,.\, Q < 650 \,.\, 1500$, a six-figure number. These

cases must be rejected since $D.Q$, the dividend, contains seven figures. Hence $a \geqslant 7$, and we are left with the following trial values of D and Q:

$$D \text{ - } \begin{cases} 741, & 742, & 743, & 744, & 746, & 747, & 748, & 749, \\ 841, & 842, & 843, & 844, & 846, & 847, & 848, & 849, \\ 941, & 942, & 943, & 944, & 946, & 947, & 948, & 949, \end{cases}$$

$$Q \text{ - } \begin{cases} 1414, & 1412, & 1418, & 1416, & 1414, & 1412, & 1413, & 1416. \\ & 1417, & & & 1419, & & 1418, & \end{cases}$$

(v) If $a=7$, $D=74c$ and $Q=141f$. The least product $D.(Q-1000)$ is $742.412 \equiv 305704$ and the greatest is $748.418 \equiv 312664$. Hence the third figure in $D.(Q-1000)$ cannot be 4, as it must from line (3), and so $a=7$ is inadmissible.

(vi) When $a=8$, $D=84c$, $Q=141f$, and the fourth line is less than $4.849 \equiv 3396$. Since the fifth line must have a 1 on the left, the third line is not less than 3540. Subtracting the greatest value of the fourth line from the least value of the third, the least value of the fifth line is 1440. Now $17.842=14314$ and $17.849=14433$, from which $f>7$. It is found that two of the three pairs of trial values remaining in (iv) give solutions, viz.:

$$846 \text{) } 1200474 \text{ (} 1419$$

and

$$848 \text{) } 1202464 \text{ (} 1418.$$

(vii) When $a=9$, the fourth line $\leqslant 4.949 \equiv 3796$ and the third $\geqslant 3940$. Hence the fifth line $\geqslant 1440$, and since $15.949=14235$, $16.949=15184$, it follows that $f \geqslant 6$. Six pairs of trial values are left in (iv), and two of them give the solutions

$$943 \text{) } 1337174 \text{ (} 1418$$

and

$$949 \text{) } 1343784 \text{ (} 1416.$$

Every possibility has now been exhausted, and there are no further solutions.

Dr. A. Clement-Jones first called my attention to this type of problem when I was a schoolboy. A few were to be found in such text-books as Workman's *Tutorial Arithmetic*. In the SEVEN SEVENS, composed *aetatis meae* xviii, and first published in the *School World* (July 1906), the data were

THE SEVEN SEVENS.

$$\begin{array}{l} \times\times\times\times 7\ \times)\times\times 7\ \times\times\times\times\times\times\times(\times\times 7\ \times\times \\ \quad \times\times\times\times\times\times \\ \quad \overline{\times\times\times\times\times 7\ \times} \\ \quad \times\times\times\times\times\times\times \\ \quad \overline{\qquad\quad \times 7\ \times\times\times\times} \\ \qquad\quad \times 7\ \times\times\times\times \\ \qquad\quad \overline{\times\times\times\times\times\times\times} \\ \qquad\quad \times\times\times\times 7\ \times\times \\ \qquad\qquad \overline{\times\times\times\times\times\times} \\ \qquad\qquad \times\times\times\times\times\times \end{array}$$

reduced to a smaller proportion than in any similar problem I had then seen—or have seen since except in positions of my own construction. Naturally it is not to be assumed that there are no sevens in the working except the ones given.

A missing-figure problem should only have a small number of solutions. A single solution is preferable unless there is some reason for allowing more than one. Four solutions to the FOUR FOURS are allowable: they would not be in the SEVEN SEVENS. Every figure given should play a vital part in the solution, and there should be just sufficient information to solve the problem, anything beyond this being cut out. Among the subjoined examples

the FIVE FIVES most nearly approaches the ideal of what a missing-figure problem should be.

THE FIVE FIVES.

```
× × × ×)× 5 5 × × 5 ×(× 5 ×
        × × 5 × ×
        -------
          × × × × ×
          × × × × ×
          ---------
              × × × ×
              × × × ×
              -------
```

The number of figures in the data can be reduced to a lower level in a division sum with no remainder than when a remainder has to be found as part of the solution. A division sum without remainder can be inverted into a multiplication, but a missing-figure problem equivalent to the latter usually has more solutions than the inverted problem in division. In the THREE

THE THREE THREES.

```
× × ×)× × 3 × ×(× ×
        × 3 ×
        -----
        × × × ×
        × × 3 ×
        -------
```

```
    × × ×
      × ×
    -----
  × × 3 ×
    × 3 ×
  -------
× × 3 × ×
```

THREES the division on the left and the multiplication on the right have each one solution. On the other hand, in the FOUR SEVENS the multiplications on the left have solutions not admitted by the divisions on the right, the reason

THE FOUR SEVENS (I).

```
    × × ×
    7 × ×
  -------
  × 7 × ×
    × × ×
    × × 7 ×
  ---------
  × × × × × 7
```

```
× × ×)× × × × × 7(7 × ×
        × 7 × ×
        -------
          × × × ×
          × × ×
          -----
          × × × ×
          × × 7 ×
          -------
```

being that the third line of each division gives a piece of information which is wanting on its left. I hope that one of these positions will satisfy Prof. H. H. Turner, who, after seeing the earlier SEVEN SEVENS and FOUR FOURS, suggested (*v. The Observatory*, July 1920) that I should propound a problem

THE FOUR SEVENS (II).

```
    × × × ×
      × × ×
    -------
  × × 7 ×
    × 7 × ×
    × × 7 × ×
  -----------
× × × 7 × × ×
```

```
× × × ×)× × × 7 × × ×(× × ×
          × × 7 ×
          -------
          × × × × ×
            × 7 × ×
            -------
            × × × × ×
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involving either four sevens or seven fours. Professor Turner will see that the unique solution to the problem on the right and the four to that on the left give the FOUR SEVENS (I) an additional reminiscent flavour of the other two. Possibly a SEVEN FOURS will be forthcoming later.

Introduction of a remainder usually requires one figure in the data for each figure in the remainder, though the figures used to determine the remainder should serve some other purpose as well. Thus, in the SEVEN SIXES the sixes in the third, fifth and seventh lines fix the remainder and also render

the solution unique. It will be found that there are quite a number of possible values for divisor and quotient when these figures are omitted.

THE SEVEN SIXES.

$$\begin{array}{l} \times\,6\,\times)\times\,6\,\times\,\times\,\times\,\times(6\,\times\,\times \\ \quad\;\times\,\times\,\times\,\times \\ \quad\overline{\quad\times\,\times\,6\,\times} \\ \quad\quad\;\times\,\times\,\times \\ \quad\quad\overline{\quad\times\,\times\,6\,\times} \\ \quad\quad\;\times\,\times\,\times\,6 \\ \quad\quad\overline{\quad\;\times\,\times\,6} \end{array}$$

The amount of data can be still further reduced when the basis of the position is a root extraction. In the TWO SIXES there are two solutions, a flaw which can just be allowed to stand. The TWO FOURS is identical with

THE TWO SIXES.

$$\begin{array}{l} \quad\times\,\times\,\times\,\times\,\times\,\times\,\times\,\times\,\times\,\times\,6(\times\,6\,\times\,\times\,\times \\ \quad\;\times\,\times \\ \times\,\times\,\times)\overline{\times\,\times\,\times\,\times} \\ \quad\quad\;\times\,\times\,\times \\ \times\,\times\,\times\,\times)\overline{\times\,\times\,\times\,\times\,\times} \\ \quad\quad\;\times\,\times\,\times\,\times \\ \times\,\times\,\times\,\times\,\times)\overline{\times\,\times\,\times\,\times\,\times\,\times} \\ \quad\quad\;\times\,\times\,\times\,\times\,\times\,\times \\ \times\,\times\,\times\,\times\,\times\,\times)\overline{\times\,\times\,\times\,\times\,\times\,\times\,\times} \\ \quad\quad\;\times\,\times\,\times\,\times\,\times\,\times\,\times \end{array}$$

the TWO SIXES, except that the sixes are replaced by fours, and that it is to be worked in scale six.

With the exception of the FOUR SEVENS (multiplication) and TWO SIXES there is just one solution to each problem in this article. All are to be regarded as problems in scale ten, except the TWO FOURS, which is in scale six. The next number of the *Gazette* will contain detailed solutions of the TWO SIXES and FIVE FIVES, together with the answers to the rest. No prizes are offered for finding the solutions.

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86. In the year 1699 Martin set out on his travels. . . In his third voyage he discovered a whole kingdom of Philosophers, who govern by the mathematicks; with whose admirable schemes and projects he returned to benefit his own dear country; but had the misfortune to find them rejected by the envious ministers of Queen Anne, and himself sent treacherously away. (This is Pope on Swift and Laputa.)—*Martinus Scriblerus*, c. ix.

87. When logarithms were invented, they were called *artificial* numbers; and the originals, for which logarithms were computed, were accordingly called *natural* numbers. Thus, in speaking of a table of sines, to express that it is not the logarithms of sines which are given, but the sines themselves, that table would be called a table of *natural* sines; and the logarithms of these would be called, not logarithms *of sines*, but logarithmic sines. *Footnote.*—This leads to confusion in the minds of students, who learn some notion of mysterious identity between a number or fraction and its logarithm; and write down $\cdot 30103 = 2$. The phrase is as incorrect as *royal country* would be for *king of the country*, or *constabulary parish* for *constable of the parish.*—De Morgan's *Trig. and Double Alg.* p. 71, 1849.

SIGN IN ELEMENTARY ANALYTICAL GEOMETRY.

By F. G. Brown, B.A., B.Sc.

The question of sign constitutes a real difficulty to the intelligent boy at the outset of his study of Coordinate Geometry. At the beginning of his Trigonometry he is told OP must be considered always positive, but later on he will find some authorities giving a point in the third quadrant as $(-r, \theta)$, while others prefer $(r, \theta+\pi)$. The perpendicular distance of (h, k) from $ax+by+c=0$ is given by $\pm(ah+bk+c)/(a^2+b^2)^{\frac{1}{2}}$, and sign seems to matter, but usually the pupil is told that he only wants to know how far off (h, k) is, and he is advised to stick to the absolute value. But a little later on he wants the equations of the bisectors of the angles between two given lines, and then he is blamed for not remembering that signs matter a good deal.

It seems advisable to have at the outset a clear distinction between terminated and non-terminated lines, the former having either one or both end-points. Thus OX, the initial line of coordinates, is terminated at O, and we agree that a movement to the right along OX is positive. Then, any line OP, terminated at both ends, and rotating about O counter-clockwise, keeps the positive sign it started with. Hence, if we draw the full rectangular axes of ordinary Cartesian geometry, OP is still positive when it lies along OX', or OY'. Then, obeying the usual conventions as to the signs of coordinates, we have, putting $OP=r=|x+iy|$,

Quadrant.		1st	2nd	3rd	4th
Signs of	x, y	$+\ +$	$-\ +$	$-\ -$	$+\ -$
By definition,	$\sin\theta=y/r$	$+$	$+$	$-$	$-$
	$\cos\theta=x/r$	$+$	$-$	$-$	$+$
	$\tan\theta=y/x$	$+$	$-$	$+$	$-$

I do not use the cumbrous forms OM/OP, etc., nor do I use the phrases "side opposite," "side adjacent."

The results tabulated above are obtained solely by means of *terminated* lines. When we enter upon a discussion of the equation of a straight line we find that $ax+by+c=0$ represents a non-terminated line.

Two cases arise:

(1) Lines parallel to the Y-axis: $x=k$.

Since the upwards direction is positive along $Y'OY$, we have no hesitation in applying the same rule to $x=k$.

(2) Lines not-parallel to the Y-axis: $y=mx+b$.

This is where the first difficulty is encountered, and I find the best way of meeting it is to extend the sign-conventions to oblique axes, the line in question being the new axis of Y. Hence "upwards" is still positive. Another way of putting it is to say that the line is to be considered as having reached its position by a positive rotation about its point of intersection with the X-axis starting from the position $X'OX$. Thus the angle of inclination with the X-axis lies between O and π; strictly

$$0\leq\theta_1<\frac{\pi}{2};\quad \frac{\pi}{2}<\theta_2<\pi;$$

θ_1 being acute, θ_2 obtuse. Hence the sine of the angle of inclination is always positive, but the tangent is positive for θ_1, negative for θ_2. Having put $x=k$ into a separate class, we need no discussion as to what happens when

$\theta=\frac{\pi}{2}$. (It should be noted that when we say "$\sin\theta$," we imply that we have made the "xyr" construction, using a terminated length of the non-terminated line as our r.)

The information "the gradient is to be m" allows me to cover the entire plane with parallel strokes.

The information "the line is to go through x_1, y_1" allows me to cover the entire plane with strokes through x_1, y_1. (Note that if I draw lines *terminated* at x_1, y_1 I can say I draw them in *all* directions, but if I draw non-terminated lines through x_1, y_1—and this is the class we wish to consider—I must restrict the angle to $0 \leq \theta < \pi$; otherwise we should cover the plane twice.)

Combining the two demands I cannot escape drawing just the one line required, and the opportunity is taken of impressing the law that the equation of the straight line contains *two* constants. If the pupil is smart enough to be told how this applies to $x=k$, then explain; otherwise you may wriggle out of the difficulty by reminding him that we have divided lines logically into two classes, and we are now dealing with lines not parallel to the Y-axis.

To "fix" a line, therefore, you must give me both (1) a fixed gradient m and (2) a fixed point (x_1, y_1). Any other point on the line may be called (x, y), and *it is impossible to deduce the relation which must exist between* $(x, y)(x_1, y_1)$ *and* m *except by equating two different ways of stating the gradient.* Once this fact is realised we shall refuse to wade through the misplaced ingenuities of the "proofs," of the ordinary text-books, for the "different forms"; all such proofs are camouflages. (Much of the difficulty felt by the beginner in this subject is due to the psychological effect of a writer's low opinion of a reader's intelligence.)

We have $\quad$ gradient $=\tan\theta=m$.

Also $\quad$ gradient $=(y-y_1)/(x-x_1)$;

$$\therefore \frac{y-y_1}{x-x_1}=m. \qquad (1)$$

I take this to be standard, although later on, partly with an eye to Solid Geometry, I write

$$\frac{y-y_1}{x-x_1}=\frac{\sin\theta}{\cos\theta}, \quad \text{and then} \quad \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\cos\left(\frac{\pi}{2}-\theta\right)}=r.$$

The next four "forms" are merely variations of (1) due to the special position of x_1, y_1.

Thus, x_1, y_1 being the origin, $\quad \frac{y}{x}=m, \qquad (2)$

x_1, y_1 being $0, b$, $\quad \frac{y-b}{x}=m, \qquad (3)$

x_1, y_1 being $a, 0$, $\quad \frac{y}{x-a}=m. \qquad (4)$

x_1, y_1 being $p\cos\alpha$, $p\sin\alpha$,

$$\frac{y-p\sin\alpha}{x-p\cos\alpha}=m=\tan\theta=\tan\left(\frac{\pi}{2}+\alpha\right)=-\cot\alpha. \qquad (5)$$

Simplify to taste. My own preference is for delay of elegance of form until the idea is firmly embedded, just as in elementary dynamics a boy should be forced to write $s=(v^2-u^2)/2a$ in order that he may be constantly reminded of the factors *average velocity* $(v+u)/2$ and *time* $(v-u)/a$. Elegant formulae appeal to the memory, but mathematical progress is quicker in the end if all elementary formulae are framed to remind the pupil, so far as may be, of the principles which were used in establishing them.

The five forms given above contain explicitly the *gradient* of the line. If the fixing of the line is made to depend on *two given points*, it is quite easy to see that m is obtainable from these two given points, and then either point is available for use in the standard form (1). Thus

$$\frac{y-y_1}{x-x_1}=\frac{y-y_2}{x-x_2}=\frac{y_2-y_1}{x_2-x_1} \quad (i.e.\ m), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

any two of which will, in fact, give the required relation, *i.e.* the equation of the straight line. In passing it may be remarked that if we are so blind as to omit to give the above in the Algebra lesson as a practical example of the beloved "k" method, then at least we should make our pupils check the above statement by straight-out multiplication. I have had many students deny offhand that the first two equated will give the correct result.

The Intercept-Form is merely a special case of (6), but I prefer to say, "Consider $x/a+y/b=1$; when x is 0, y is b; when y is 0, x is a; therefore, etc."

Of course we must include all the forms under the general $ax+by+c=0$, and we must on no account omit to discuss the availability of the equation for every point on the line, and the existence upon the line of every point whose coordinates satisfy the equation.

Now, as regards direction, the advantage of the standard form (1), and of deriving all others from it, lies in the fact that on the left-hand side we have a *ratio*, and it is immaterial whether we write

$$\frac{y-b}{x-0}=m \quad \text{or} \quad \frac{b-y}{0-x}=m.$$

To the beginner this is no small matter, and he should be practised in deriving the standard equation with the fixed point in all quadrants and for all slopes (except $\theta=\pi/2$).

We already know that the Y-axis divides the whole plane, so that measurements *from the axis* and parallel to the X-axis are positive if to the right, negative if to the left. Drawing a line obliquely across the X-axis we follow exactly the same rule. Hence the distance, parallel to the X-axis, *from* $Q(x_2, y_1)$ on the line $ax+by+c=0$ *to* $P(x_1, y_1)$ is x_1-x_2 and is always positive (negative) when P is to the right (left) of the given line.

A limiting case arises when the given line is the X-axis itself, or is parallel to it, but in this case there can be no right nor left-hand side to the line. We can merely say that for any two points on such a line a measurement to the right (left) is pos. (neg.). Just as in studying the equation of the straight line we considered it expedient to put all lines whose slope is infinite into a separate class, so now we have a separate class whose slope is zero, and any member of this class divides the plane into *upper* and *lower*, instead of left and right.

Line.	In the equation, substitute	Resulting sign of L.H.S. of equation.	Therefore,
$y=0$	x_1 = anything. y_1 = pos. number.	+	Upper side +
$y=0$	x_1 = anything. y_1 = neg. number.	−	Lower side −
$y-2=0$	x_1 = anything. $y_1=4$.	+	Upper side +
$y+3=0$	x_1 = anything. $y_1=-4$.	−	Lower side −

Having thus disposed of all lines whose slope is zero, we deal with the general equation $ax+by+c=0$, excluding only the class thus disposed of.

Let x_2, y_1 be a point on the line.

Then $$ax_2+by_1+c=0.$$

Let x_1, y_1 be a given point not on the line.

Then $$\begin{aligned} ax_1+by_1+c &= ax_1+by_1+c-0 \\ &= ax_1+by_1+c-(ax_2+by_1+c) \\ &= a(x_1-x_2). \end{aligned}$$

Now, if x_1, y_1 is to the right of the given line, x_1-x_2 is positive, and if we take care to have a positive when writing $ax+by+c=0$, we shall then ensure that the substitution of x_1, y_1 in the left side of this equation will lead to a positive result, whereas x_1-x_2 is negative if x_1, y_1 is to the left, and with the same precaution as to the sign of a we shall get the desired negative result. (Another device is to divide through by a so as to leave the coefficient of x positive unity, as is usually done in the first step of the solution of the general quadratic.) In this way there can be no ambiguity whatever as to which side of a given line is positive.

The perpendicular from $ax+by+c=0$ *to the point* x_1, y_1.

Take, as before, x_2, y_1 a point on the line.

Then $$x_2=-\frac{by_1+c}{a}$$

and $$x_1-x_2=\frac{ax_1+by_1+c}{a}.$$

Now $p/(x_1-x_2)=\sin\theta$, where θ is the angle of inclination of the line with the X-axis, and, in our system, θ is always positive and less than π, hence $\sin\theta$ is always positive.

But $\tan\theta=-a/b$; hence $\sin\theta=$ the absolute value of $a/\sqrt{(a^2+b^2)}$.

Hence $$p=(x_1-x_2)\sin\theta=\frac{ax_1+by_1+c}{a}\cdot\frac{a}{\sqrt{(a^2+b^2)}}.$$

When $(ax_1+by_1+c)/a$ is $+$, p is $+$, because $\sin\theta\,(=a/\sqrt{(a^2+b^2)})$ is $+$.
" " " $-$, p is $-$, " " " "

In brief, the sign of p (the perpendicular *from* the line *to* the point) is entirely dependent on the sign of $(ax_1+by_1+c)/a$, *i.e.* of x_1-x_2.

Thus, p from $2x+3y+6=0$ to 1, $1=\dfrac{2+3+6}{2}\cdot\dfrac{2}{\sqrt{13}}$ (pos.),

" from $-2x+3y+6=0$ to 1, $1=\dfrac{-2+3+6}{-2}\cdot\dfrac{2}{\sqrt{13}}$ (neg.)

(if preferred, write $\dfrac{-2}{-\sqrt{13}}$ in this last case),

" from $2x-3y+6=0$ to 1, $1=\dfrac{2-3+6}{2}\cdot\dfrac{2}{\sqrt{13}}$ (pos.),

" from $2x+3y-6=0$ to 1, $1=\dfrac{2+3-6}{2}\cdot\dfrac{2}{\sqrt{13}}$ (neg.),

" from $-2x-3y-6=0$ to 1, $1=\dfrac{-2-3-6}{-2}\cdot\dfrac{2}{\sqrt{13}}$ (pos.).

On plotting the lines it will be found that p in every case conforms to the right-or-left-hand-side-of-the-line convention.

Also in every case it will be found that p from the line to the origin (0, 0) will have the same sign as any other p from the line to a point on the same side of the line as the origin.

This last result may cause some difficulty, since p is a terminated line, and we appear to have agreed that a vector rotating about the origin shall remain positive in all positions. But there is no real contradiction. Our rule was laid down to enable us to define the trigonometrical ratios, and, in so far as these ratios have to do with the determination of p, we are concerned only with the ratios of θ—the slope of the line $ax+by+c=0$ and not the slope of p. The difficulty occurs again when we use Hesse's Normal Form (where "Normal" is so often misinterpreted as "perpendicular" instead of "standard" or "regular")

$$x\cos\alpha+y\sin\alpha=p.$$

Here we *are* concerned with the angle (α) which p traverses, starting from its initial position along OX, and hence we must consider p essentially positive. Furthermore the angle α is not confined within the limits we imposed for θ, the angle of slope of a non-terminated line, but is able to take all values. The form should be introduced by showing that we can "fix" a line if we agree it shall be at right angles to, and at the outer end of, a given spoke.

The only excuse for the early introduction of Hesse's Form into most standard texts seems to be its utilisation for the finding of the perpendicular distance from (to) a line to (from) a given point. My own preference is for delay, until higher powers of analysis enable the pupil to use the form in connection with the circle. It seems bad pedagogy to use Hesse for the fundamental result, then to formulate a memory-rule whereby this result can be applied to $ax+by+c=0$, and then to use this latter rule for the determination of the equations of the bisectors of "the angles between two lines."

The Equation of the Bisector of the Angle between two given lines.

Let $$A=ax+by+c=0$$

and $$B=dx+ey+f=0$$

be the given lines, and suppose that the gradient of B is greater than that of A. Then *the* angle between the lines is the positive angle through which B has turned from coincidence with A, and it will always be found that if x_1, y_1 be a point on the bisector of this angle, then the perpendiculars *from* A and B respectively *to* the point x_1, y_1 will have *opposite* signs. These perpendiculars are equal in absolute value. Hence we have

$$\frac{ax_1+by_1+c}{a}\cdot\frac{a}{\sqrt{(a^2+b^2)}}=-\frac{dx_1+ey_1+f}{d}\cdot\frac{d}{\sqrt{(d^2+e^2)}},$$

and since this is true for any point x_1, y_1 on the bisector we have the equation of the bisector by dropping the subscripts.

From the Trigonometrical point of view the angle between two non-terminated lines is multivalued; from the Euclidean point of view there are four angles; for the purposes of Analytical Geometry there is only one angle.

One of the lines has rotated θ_1, the other θ_2, and the angle between them is $\theta_1 \sim \theta_2$. I always insist on an inspection of the given equations to find which is the greater angle; then this angle (say θ_1) is the exterior angle of a triangle in which one of the interior angles is θ_2, or is vertically opposite to θ_2, and the other is *the* angle between the lines.

The bisector of the supplementary angle between the lines is of course given by equating the same expressions as above but with *like* signs, and it is a useful problem to discover why this must always be.

Ex. $$x+3y-6=0,$$

$$3x+y+2=0.$$

The bisector is $$\frac{x+3y-6}{1}\cdot\frac{1}{\sqrt{10}}=-\frac{3x+y+2}{3}\cdot\frac{3}{\sqrt{10}}$$

or $$x+y-1=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The bisector of the supplementary angle is

$$x - y + 4 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and these are "the bisectors of the angles between, etc." Examiners often ask a candidate to "distinguish" between the bisectors, and there are wordy rules about the bisector of "the angle containing the origin." The above method contains a truer distinction.

It is not claimed that the conventions indicated in this brief paper are entirely satisfactory; Bôcher in his *Plane Analytic Geometry* indicates that neither his own convention "nor any other one which could be made" can be satisfactory. But I prefer to take $\sqrt{(a^2+b^2)}$ as being always unambiguous and to make the sign of a perpendicular *from* a line *to* a point depend solely upon the value of $x_1 - x_2$ (*i.e.* abscissa of point *minus* the abscissa of the point on the line where $y = y_1$ cuts it). The use of $\sin\theta$ makes this possible. It will be noted too that the perpendicular from $y = 2$ to (3, 0) turns out to be negative, as it should, and the perpendicular from $y = 0$ to (1, 1) is positive, but the application of my rules to these cases introduces the form 0/0, and it is best to follow the ordinary rule of signs, familiar from graph-work, with the comforting assurance that these special cases do not form exceptions, nor do we get a sudden change of value on proceeding to the limit.

Immediately you try to frame satisfactory rules for the perpendicular *from a point* to a line you run up against the fundamental convention of Cartesians, which requires (x, y) to be arrived at by travelling from the axes and not towards them.

Sydney. F. G. BROWN.

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88. The American Cocker—"I have heard devils can be raised with Daboll's arithmetic. . . . That's my small experience as far as the Massachusetts calendar, and Bowditch's navigator, and Daboll's arithmetic go."—p. 515, Herman Melville's *Moby Dick* (World's Classics).

89. Tautochronism.—"It was in the left-hand try-pot of the Pequod, with the soapstone diligently circling round me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time."—Herman Melville, *Moby Dick*, p. 502.

90. Sir Isaac Newton, it is said, spoke with much contempt (but surely without just grounds) of those two accomplished scholars and critics (Bentley and Hare) for squabbling (as he expressed it), about an old play-book (Terence). —[Whiston mentions this in his *Memoirs of Dr. Clarke*: Warton, Note to *Martinus Scriblerus*, c. ix.]

91. It was said of the first mathematicians that they opened a field in which their successors go on advancing and behold the horizon receding at every step. He who enters on this fair field must be ever pressing forward and consider nothing as done while anything remains undone.—Graves' *Hamilton*, i. 112.

92. Captain Cook.—Christie's sale last week included an important MS. of Captain James Cook, "Arithmetical Trigonometry" and "Arithmetical Dialling," 1763, the property of the late Mr. H. H. Emson, for which Messrs. Quaritch paid £200. The MS. extends to 97 pages folio, and contains elaborately and beautifully drawn diagrams, which bear out the statement in the *Dictionary of National Biography* that Captain Cook might "under other circumstances have proved himself as eminent as a surveyor as he actually did as an explorer." He was also something of a versifier, for he has proclaimed his ownership of this MS. in two verses on the first page. Some MSS. of Captain Cook were sold at Puttick and Simpson's on March 10, 1868, and this may have been among them.—*Times Lit. Supp.*, Aug. 4, 1921.

MATHEMATICAL NOTES.

590. [L¹. 15. b.] *A Note on Inversion.*

The figure represents a parabola, focus S, directrix XM.

The circle on PM as diameter cuts the focal radius SP at J. SEF is a diameter of this circle.

The triangles SPN, PMJ are congruent, so that $JP=SN$; $\therefore$ $SJ=SX$.

Invert the figure with respect to S.

The directrix becomes a circle on SX' as diameter.

The circle PJM becomes a circle, whose diameter is $E'F'$.

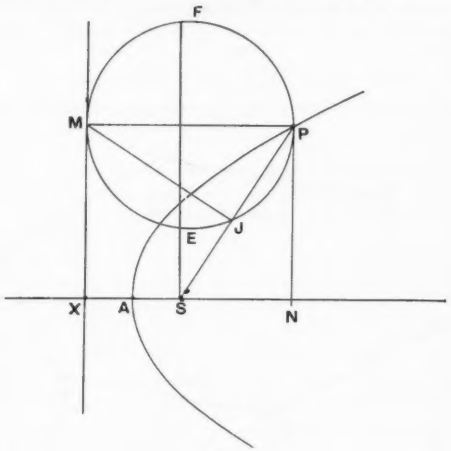


FIG. 1.

Now $$E'F'=\frac{k^2EF}{SE\,.\,SF}=\frac{k^2PM}{SJ\,.\,SP}=\frac{k^2}{SX}=SX'.$$

Hence, for all positions of P the circle on PM as diameter inverts into a circle *equal in magnitude* to the inverse of the directrix and touching it.

Further, $$P'M'=\frac{k^2PM}{SP\,.\,SM}=\frac{k^2}{SM}=SM'.$$

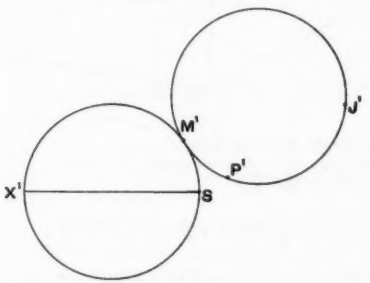


FIG. 2.

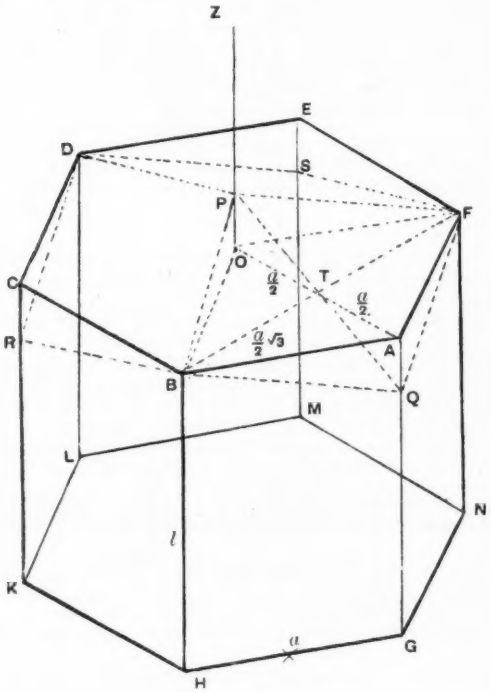
Hence, since the circles are equal, and the chords SM', $P'M'$ are equal, the arcs SM', $P'M'$ are equal. It follows that the locus of P' coincides with that of a point fixed on a circle equal to and rolling on $SM'X'$.

This proof obtains the epicycloid definition of the cardioide by inversion from the parabola. A. CLEMENT JONES.

591. [K¹. 14.] *Elementary Solution of Bee Cell Problem.*

Given (i) cubic content to equal that of prism terminated by the two regular hexagons $ABCDEF$, $GHKLMN$, of length l.

(ii) Cell to be closed at one end by 3 equal rhombs $FQBT$, $BRDP$, $DSFP$ meeting at a point P on the axis OZ of the prism.



(iii) The surface to be a minimum.

Let a be a side of one of the hexagons.

Surface $= 6(BPQ + BQGH)$

$$= 6\left(\frac{1}{2}\cdot\frac{a}{2}\sqrt{3}.2\frac{a}{2}\operatorname{cosec}\phi + la - \frac{1}{2}a\frac{a}{2}\cot\phi\right)$$

$$= 6la + \frac{3a^2}{2}(\sqrt{3}\operatorname{cosec}\phi - \cot\phi),$$ where ϕ = inclination of rhombs to axis.

Put $$y=\sqrt{3}\operatorname{cosec}\phi-\cot\phi.$$

$$2\cot^2\phi-2y\cot\phi+3-y^2=0,$$

$$\cot\phi=\frac{y\pm\sqrt{3y^2-6}}{2};$$

$$\therefore \text{ min. value of } y^2=2, \text{ giving } \cot\phi=\frac{1}{\sqrt{2}};$$

$$\therefore AQ=\frac{a}{2\sqrt{2}};\ \tan AQB=2\sqrt{2};\ \cos AQB=\frac{1}{3};$$

$$\therefore BQ=\frac{3a}{2\sqrt{2}};\ \cos TBQ=\frac{\sqrt{2}}{\sqrt{3}};\ \cos PBQ=\frac{4}{3}-1=\frac{1}{3}=\cos AQB;$$

$$\therefore P\hat{B}Q=A\hat{Q}B \quad\text{and}\quad B\hat{Q}F=B\hat{Q}G=F\hat{Q}G.$$

Founded on Maclaurin's geometrical proof given at some length in Cresswell's *Maxima and Minima*. E. M. Langley.

592. [L¹. 5. c.] *To find the evolute of the conic in* (1) *Cartesians*, (2) *trilinears.*

Eliminant of $a\beta\gamma$ between $L=0$, $L'=0$ and $S=0$ is $\Sigma\Sigma'-\Pi^2=0$.

Cartesians. If (x_1y_1) is the point at which the line $lx+my+n=0$ is normal, the tangent is $xX_1+yY_1+Z_1=0$;

$$\therefore lX_1+mY_1 \quad\text{or}\quad (al+hm)x_1+(hl+bm)y_1+(gl+fm)=0.$$

Calling this $l'x_1+m'y_1+n'=0$, we have to eliminate x_1y_1 between this,

$$lx_1+my_1+n=0 \quad\text{and}\quad S_1=0.$$

Now $$\Sigma'=l'(Al'+Hm'+Gn')+\ldots+\ldots$$
$$=l'\Delta l+m'\Delta m=\Delta(al^2+2hlm+bm^2),$$

and $$\Pi=l.\Delta l+m.\Delta m=\Delta(l^2+m^2);$$

$\therefore$ tangential equation of evolute is $(al^2+2hlm+bm^2)\Sigma=\Delta(l^2+m^2)^2$.

Trilinears. Here $(\alpha_1\beta_1\gamma_1)$ is the point of contact and $l\alpha+m\beta+n\gamma$ is to be perpendicular to $\alpha X_1+\beta Y_1+\gamma Z_1=0$;

$$\therefore lX_1+mY_1+nZ_1-(mZ_1+nY_1)\cos A$$
$$-(nX_1+lZ_1)\cos B-(lY_1+mX_1)\cos C=0;$$

or $$pX_1+qY_1+rZ_1, \text{ where } p=l-m\cos C-n\cos B, \text{ etc.};$$

or $$l'\alpha_1+m'\beta_1+n'\gamma_1=0,$$

where $l'=ap+hq+gr$.

Now $$\Pi=l(Al'+Hm'+Gn')+\text{etc.}$$
$$=\Delta(pl+qm+rn)=\Delta\Omega,$$

and $$\Sigma'=l'(Al'+Hm'+Gn')+\text{etc.}$$
$$=(ap+hq+gr)\Delta p+\ldots$$
$$=\Delta(ap^2+bq^2+cr^2+2fqr+2grp+2hpq)$$
$$=\Delta H\Omega-\Sigma P+\overline{K}^2,$$

where $$H=a+b+c-2f\cos A-2g\cos B-2h\cos C,$$
$$P=A\sin^2A+\ldots+2F\sin B\sin C,$$
$$\overline{K}=\Sigma(Al+Hm+Gn)\sin A;$$

$\therefore$ evolute is $$\Sigma(\Delta H\Omega-\Sigma P+\overline{K}^2)=\Delta^2\Omega^2,$$

or $$\Delta^2\Omega^2-\Delta H\Sigma\Omega+P\Sigma^2=\Sigma\overline{K}^2,$$

i.e. $$[\text{Foci}]=[\text{Conic}][\text{Centre}]^2.$$

N. M. Gibbins.

593. [U.] *Note on the Equation of Time.*

(1) An elementary course on Astronomy forms part of the curriculum of many schools. In this connection the Equation of Time forms such a good example on compounding two sine graphs that it seems worth while to suggest an elementary way of approaching it. The main difficulty in arriving at the formula in an elementary way is due to the part of the Equation of Time that depends on the unequal motion in the Ecliptic. The part due to the obliquity of the Ecliptic to the Equator requires only a little Spherical Trigonometry, and it is given below in the usual way. An attempt is here made to get the former part on the assumption that the orbit of the Earth is a circle, the Sun's position being slightly eccentric. The only other assumption made is Kepler's Law on the equable description of areas.

(2) The following are the usual definitions, which are collected together for purposes of reference.

The *Apparent Sun* (actual Sun) moves in the Ecliptic with non-uniform angular velocity.

The *Dynamical Mean Sun* is an imaginary heavenly body that moves in the Ecliptic with the Apparent Sun's mean angular velocity and which coincides with the Apparent Sun at Perigee and Apogee.

Mean Time is the time that would be kept by an imaginary heavenly body called the Mean Sun.

The *Mean Sun* is supposed to move in the Equator in such a manner that its Right Ascension is equal to the Longitude of the Dynamical Mean Sun.

The *Equation of Time* is the correction to be added to Apparent Time to obtain Mean Time; or it may be defined as the quantity to be added to the Longitude of the Dynamical Mean Sun to give the Apparent Sun's R.A. The two definitions may be reconciled as follows:

Let α = R.A. of the App. Sun,

L = Long. of Dyn. Mean Sun.

At any Sidereal Time T, the Hour Angle of the App. Sun, $T-\alpha$;

and the Hour Angle of the Mean Sun, $T-L$;

also $$\alpha - L = (T-L) - (T-\alpha)$$

$$= \text{Mean Time} - \text{App. Time}.$$

(3) The next step is to express α and L in terms of the Long. of the App. Sun, which in the following is denoted by l.

It will be shown that

$$l - \alpha \doteqdot 9{\cdot}9 \sin 2l \text{ minutes},$$

$$L - l \doteqdot 7{\cdot}7 \sin (l+79^\circ) \text{ minutes},$$

so that the Equation of Time for any Long. l of the App. Sun is

$$\doteqdot 7{\cdot}7 \sin (l+79^\circ) - 9{\cdot}9 \sin 2l \text{ minutes}.$$

The former part is due to the unequal motion in the Ecliptic and the latter part to the obliquity of the Ecliptic to the Equator.

(4) To find the part due to the unequal motion. The semi-diameter of the Sun and his Longitude are given in the *Nautical Almanac* for every day of the year at Mean Noon. From these the shape of the Earth's orbit may be plotted. The values for Perigee and Apogee, when the Sun is nearest and farthest from the Earth, and for the two dates whose Longitudes differ from these by 90°, are as follows:

	Long.	Semi-diam.	Reciprocal of Semi-diam.
Jan. 3 (perigee) - - -	281° 50′	977·57″	$1{\cdot}0230 \times 10^{-3}$
April 1 - - - -	11° 28′	961·49″	$1{\cdot}0400 \times 10^{-3}$
July 4 (apogee) - - -	101° 13′	945·37″	$1{\cdot}0578 \times 10^{-3}$
October 5 - - - -	191° 56′	961·56″	$1{\cdot}0400 \times 10^{-3}$

The radii vectores from the centre of the Sun to the centre of the Earth on these four days are therefore proportional to the numbers in the last column. In Fig. 1 S is the Sun, A_1, A_2, A_3, A_4, the positions of the Earth on these four days.

Also
$$\frac{A_1A_3}{A_2A_4}=\frac{10230+10578}{10400+10400}=\frac{20808}{20800}=1+\frac{4}{10400}.$$

Thus the length of the chord A_2A_4 differs from the length of the diameter A_1A_3 by less than 0·04 per cent., and the diameter through C (the middle point of A_1A_3), at right angles to A_1A_3, will of course differ from A_1A_3 by a still smaller percentage.

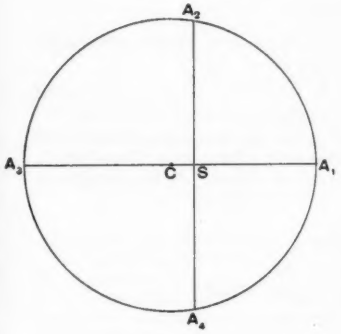


FIG. 1.

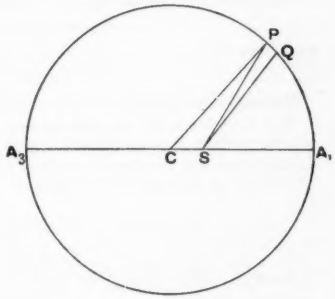


FIG. 2.

The orbit of the Earth may thus be taken to be circular to a very close degree of approximation.

If we put $A_1A_3=2a$ and $CS=ae$,

then
$$a+ae=A_3S=10578$$

and
$$a-ae=A_1S=10230;$$

therefore
$$e=\frac{10578-10230}{10578+10230}=\frac{348}{20808}=0{\cdot}0167=\frac{1}{60}\text{ nearly.}$$

This gives the eccentricity of the Earth's orbit. If then the orbit is represented by a circle of radius 10 inches, the Sun will be at a distance 0·167 inch from the centre of the circle.

(5) In Fig. 2 let P be the position of the Earth in its orbit at any instant. Suppose a body Q to revolve in the Earth's orbit with the Earth's mean angular velocity about S. Let Q in the figure be the position of this body when the Earth is at P, and suppose that P and Q coincide at A_1 and therefore also at A_3.

Denote the angle A_1SP by v, the angle A_1CP by u, and the angle A_1SQ by m. Now Kepler's Second Law states that the time of describing the arc A_1P is proportional to the area A_1SP, and therefore

$$\frac{\text{area } A_1SP}{\pi a^2}=\frac{\text{angle } QSA_1}{2\pi} \quad \text{or} \quad \frac{\text{area } A_1SP}{\pi a^2}=\frac{m}{2\pi};$$

$$\therefore \text{ area } A_1SP=\frac{m}{2}a^2.$$

But area A_1SP = area A_1CP − area CSP;

$$\therefore \frac{ma^2}{2} = \frac{ua^2}{2} - \frac{a^2e}{2}\sin u$$

or
$$m = u - e\sin u. \quad \text{.....(1)}$$

From the triangle CSP,

$$\sin(v-u) = \frac{ae}{a}\sin v = e\sin v.$$

Now $v-u$ has its maximum value when $v = 90°$;

$$\therefore \sin(v-u) \gtrdot e \gtrdot 0{\cdot}0167;$$

$$\therefore v - u \gtrdot 57'.$$

Therefore, to a first degree of approximation,

$$v - u \doteqdot e\sin v; \quad \text{.....(ii)}$$

and from (i)
$$m \doteqdot v - e\sin v - e\sin u$$
$$\doteqdot v - 2e\sin v. \quad \text{.....(iii)}$$

We can now transpose this equation into longitudes.

The Longitude of the Sun as seen from P is the Apparent Longitude l, and the Longitude as seen from Q is the Longitude of the Dynamical Mean Sun L. Longitudes are measured from γ (First Point of Aries). If the Longitude of Perigee is denoted by λ, then

$$v + \lambda = l,$$
$$u + \lambda = L;$$
$$\therefore L \doteqdot l - 2e\sin(l-\lambda).$$

Now $\lambda \doteqdot 281°$, see § (1), and $\sin(l - 281°) = \sin(79° + l)$;

and therefore
$$L = l - 2e\sin(79° + l). \quad \text{.....(iv)}$$

(6) It is only necessary now to get the part of the equation of time due to the obliquity of the Ecliptic.

In Fig. 3 γM is the Equator, γS is the Ecliptic, S is the Apparent Sun and the angle $SM\gamma = 90°$, so that $\gamma M = \alpha$ and $\gamma S = l$.

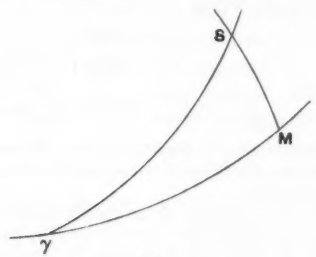


FIG. 3.

Denoting the angle $S\gamma M$ by ω (obliquity of the Ecliptic), we have in the right-angled spherical triangle γMS,

$$\cos\omega = \frac{\tan\alpha}{\tan l}; \quad \text{.....(v)}$$

$$\therefore \tan^2\frac{\omega}{2} = \frac{1-\cos\omega}{1+\cos\omega} = \frac{\tan l - \tan\alpha}{\tan l + \tan\alpha} = \frac{\sin(l-\alpha)}{\sin(l+\alpha)};$$

$$\therefore \sin(l-\alpha) = \tan^2\frac{\omega}{2}\sin(l+\alpha). \quad \text{.....(vi)}$$

The maximum value of $\sin(l-\alpha)$ is therefore given when $l+\alpha=90°$, and is $=\tan^2\frac{\omega}{2}$. Since $\omega=23°\ 27'$, it is easy to show that the maximum value of $l-\alpha$ is $2°\ 28'$, and therefore l differs from α by less than $2\frac{1}{2}°$.

If we replace α by l in the right-hand side of equation (vi), we get

$$\sin(l-\alpha) \doteqdot \tan^2\frac{\omega}{2}\sin 2l.$$

It may be shown that the maximum error in $l-\alpha$ produced by this assumption is less than 4 per cent.

Therefore, to a first degree of approximation,

$$l-\alpha \doteqdot \tan^2\frac{\omega}{2}\sin 2l. \qquad \text{.....(vii)}$$

(7) From equations (iv) and (vii) we get immediately

$$\alpha - L \doteqdot 2e\sin(79°+l) - \tan^2\frac{\omega}{2}\sin 2l. \qquad \text{.....(viii)}$$

This gives the Equation of Time in radians. If we express it in minutes of time, we get

$$2e \text{ radians} = 2\times 0{\cdot}0167\times 57{\cdot}3\times 4 \doteqdot 7{\cdot}7 \text{ minutes},$$

$$\tan^2\frac{\omega}{2} \text{ radians} = (0{\cdot}207)^2\times 57{\cdot}3\times 4 \doteqdot 9{\cdot}9 \text{ minutes}.$$

Therefore the Equation of Time $\doteqdot (7{\cdot}7\sin(79°+l) - 9{\cdot}9\sin 2l)$ minutes.

The Longitude of the Sun increases about 1° per day, so that if a *Nautical Almanac* is not at hand a fairly accurate graph may be obtained by supposing the year to consist of 12 months each of 30 days. The most rapid and most instructive way of obtaining the final graph is to plot the graphs of $7{\cdot}7\sin(79°+l)$ and $9{\cdot}9\sin 2l$, and to obtain the final graph by measuring off the differences of the ordinates.

R. M. MILNE.

594. [R. 5. a.] *A Spherical Shell of Gravitating Matter acts at an External Point as if its Mass were concentrated at the Centre.*

Draw a section through the centre C of the sphere, and let A be the external point, distant a from the centre.

Consecutive secants AP_1 and AP_2 through A cut off arcs ds_1 and ds_2, which generate corresponding zones on rotating the figure about AC.

Let Q be the pole of P_1P_2, and let P_1P_2 (P_1, P_2 on opposite sides of CQ) subtend an angle 2θ at the centre, $C\hat{A}P$ being denoted by ϕ.

Areas of zones are $2\pi\,.\,AP_1\sin\phi\,.\,AP_1\,d\phi\ \operatorname{cosec}\theta$

and $2\pi\,.\,AP_2\sin\phi\,.\,AP_2\,d\phi\ \operatorname{cosec}\theta$.

Component of attraction along CA is

$$dF = \sigma\,.\,4\pi\sin\phi\cos\phi\ \operatorname{cosec}\theta\,d\phi,$$

where σ is the surface density.

But $$r\cos\theta = a\sin\phi,$$

so that $$-r\sin\theta\,d\theta = a\cos\phi\,d\phi.$$

Hence $$dF = -4\pi\sigma\frac{r^2}{a^2}\cos\theta\,d\theta,$$

i.e. $$F = -\frac{4\pi\sigma r^2}{a^2}\int_{\frac{\pi}{2}}^{0}\cos\theta\,d\theta = \frac{4\pi\sigma r^2}{a^2} = \frac{M}{a^2}.$$

COR. *A sphere described on AC as diameter divides the attracting sphere into two parts, which exert equal attractive forces at A.*

GUY T. P. TATHAM.

The University, Leeds.

595. [L¹. 4. a.] *Proof that the Tangent to a Conic is equally inclined to the Focal Vectors.*

The following geometrical proof is extremely simple.

1. Let F and G be two points on opposite sides of a line l, which meets FG at M. Then, if P and Q are two points on l such that $FP+PG=FQ+QG$, P and Q are on opposite sides of M.

Let F' be the image of G in l; it follows that, if F and F' are two points on the same side of l, and P and Q are two points on l such that

$$FP+PF'=FQ+QF',$$

there is a point M on l between P and Q such that MF and MF' are equally inclined to PQ. The conditions hold if P and Q are any two points on an ellipse (both on the same side of the major axis), F and F' being the foci. Making Q tend towards P, we get the theorem for an ellipse.

2. The proof for a hyperbola is almost identical. Take P and Q on the same branch and on the same side of the transverse axis. The lemma required is this: If F and G are on the same side of l, and P and Q on l are such that $FP-PG=FQ-QG$, P and Q are on opposite sides of the intersection M of l and FG.

M. F. EGAN.

35 Lower Leeson Street, Dublin.

596. [K¹. 1. d.] *Remarks on Note* 554: *Area of a Triangle in Terms of the Coordinates of its Angular Points.*

It seems possible to meet Austral's wishes even more determinantly.

Let $A(0,0)$, $B(x_1,y_1)$, $C(x_2,y_2)$ be the angular points of the triangle.

In polars, B is (r_1,θ_1), C is (r_2,θ_2).

$$\text{The area} = \tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)$$

$$= \tfrac{1}{2}r_1r_2(\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1)$$

$$= \tfrac{1}{2}r_1r_2\begin{vmatrix}\sin\theta_2 & \sin\theta_1\\ \cos\theta_2 & \cos\theta_1\end{vmatrix}, \quad \text{by the fundamental definition of a determinant-form,}$$

$$= \tfrac{1}{2}\begin{vmatrix}r_2\sin\theta_2 & r_1\sin\theta_1\\ r_2\cos\theta_2 & r_1\cos\theta_1\end{vmatrix}, \quad \text{by the common-factor rule,}$$

$$= \tfrac{1}{2}\begin{vmatrix}y_2 & y_1\\ x_2 & x_1\end{vmatrix}$$

$$= \tfrac{1}{2}\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}.$$

With angular points $A(x_3,y_3)$, $B(x_1,y_1)$, $C(x_2,y_2)$, A being (r_3,θ_3),

$$\text{the area} = OBC+OCA+OAB, \text{ in all cases,}$$

$$= \tfrac{1}{2}\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}+\tfrac{1}{2}\begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}+\tfrac{1}{2}\begin{vmatrix}x_3 & y_3\\ x_1 & y_1\end{vmatrix}$$

$$= \tfrac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}.$$

On the question of sign, follow the usual convention,—positive area is on your left as you traverse its boundary. "Austral's" method uses the absolute value of the perpendicular from a point to a line, and with numerical values his result may be positive or negative. The above method avoids this ambiguity. It also helps to prevent the pupil from concluding that you never get determinants except by elimination when using equations. One gets the same sort of fixed idea in connection with S.H.M. and the circle.

Sydney. F. G. BROWN.

597. [X. 4.] *Geometrical Construction for finding a number x, such that $\frac{1}{x}=\frac{1}{a}+\frac{1}{b}$, where a and b are known.*

Draw two lines OA, OB cutting at 120° of lengths a, b. Let the bisector of the angle AOB meet AB in X. Then $OX=x$.

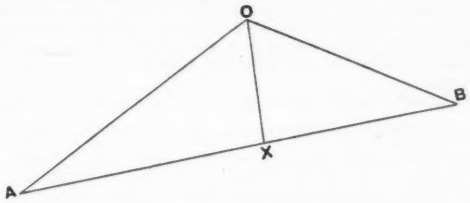


FIG. 1.

This can be extended as follows:
Given r_0, r_1, r_2, ... to find R_n such that

$$\frac{1}{R_n}=\frac{1}{r_0}+\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}.$$

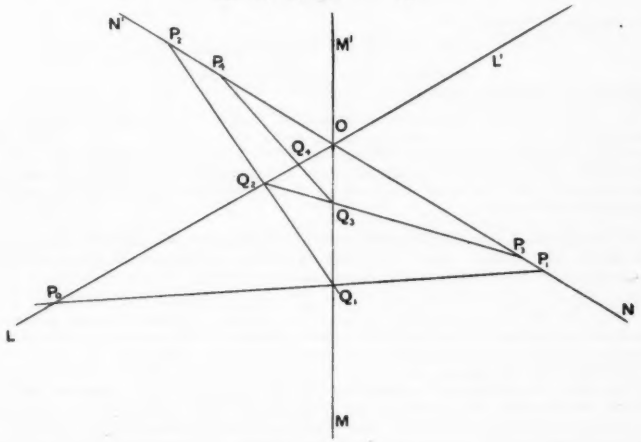


FIG. 2.

Take 3 lines LOL', MOM', NON' such that $\hat{LON'}=\hat{LOM'}=\hat{MON}=60°$.

Along OL mark a point P_0 such that $OP_0=r_0$; along ON, mark points P_1, P_3, P_5, ..., and along ON', mark points P_2, P_4, P_6 ..., such that $OP_1=r_1$, $OP_2=r_2$, etc. Let P_0P_1 cut OM in Q_1, Q_1P_2 cut OL in Q_2, Q_2P_3 cut OM in Q_3, and so on. Then $OQ_n=R_n$.

If any one (or more) of the quantities r (say r_k) is negative, then OP_k must be reversed in direction,

i.e. P_k will lie in ON' if k is odd and in ON if k is even;

and again, if any of the points Q (*e.g.* Q_m) lie on OL' or OM', the corresponding value of R (R_m) will be negative.

The College, Dartmouth. D. F. FERGUSON.

REVIEWS.

An Elementary Text-Book of Trigonometry. By R. W. K. EDWARDS. 5s. net. Pp. xiii+251. 1921. (Harrap.)

This book is a revised edition of one published in 1910, but very many errors remain uncorrected. It is an easy text-book, written for the most part on familiar old-fashioned lines, and intended primarily for candidates preparing for the Intermediate examinations of the University of London; and it contains a large number of exercises, the answers to which are given at the end of the book.

After dealing with the measurement of angles, the six trigonometrical ratios are introduced together in Chapter III., the mnemonic, Some $\frac{\text{People}}{\text{Have}}$ Curly $\frac{\text{Brown}}{\text{Hair}}$ Till $\frac{\text{Painted}}{\text{Black}}$, suggesting Sin $\frac{\text{Perp.}}{\text{Hyp.}}$ Cos $\frac{\text{Base}}{\text{Hyp.}}$ Tan $\frac{\text{Perp.}}{\text{Base}}$, being given as a help to beginners, with the alternative "sin perplexes hypocrites" as a mnemonic for the *sine*. Would it not be better to introduce the *cosine* only, and let beginners learn its meaning and use before proceeding to the other functions?

There is a startling statement on page 200: From the two identities

$$\left.\begin{aligned}\cos\tfrac{1}{2}A+\sin\tfrac{1}{2}A&=\pm\sqrt{1+\sin A}\\ \cos\tfrac{1}{2}A-\sin\tfrac{1}{2}A&=\pm\sqrt{1-\sin A}\end{aligned}\right\}$$

are deduced the two

$$\left.\begin{aligned}2\sin\tfrac{1}{2}A&=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\\ 2\cos\tfrac{1}{2}A&=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}\end{aligned}\right\},$$

whence, it is stated, $\sin\frac{1}{2}A=\cos\frac{1}{2}A$.

Towards the end of the book there is a short chapter on Hyperbolic Functions, denoted by hin x, hos x, han x, hot x, hec x and hosec x. The reader will probably wonder why hin x is defined as $\frac{1}{2}(e^x-e^{-x})$ rather than $\frac{1}{2}(10^x-10^{-x})$, especially as the table of formulae given on pages 227 and 228 would remain unchanged, while the table of common antilogarithms would be available for numerical calculation. W. J. D.

Bibliotheca Chemico-Mathematica: Catalogue of Works in many Tongues on Exact and Applied Science, with a Subject Index. Compiled and annotated by H. Z(EITLINGER) and H. C. S(OTHERAN). Two vols. Pp. xii+964, with 127 plates. £3. 3s. 1921. (Sotheran and Co.)

While we should be the last to deny the charm of these volumes, not even the many delightful hours we have spent with Mr. Sotheran's catalogues and among his books can induce us to regard the work, judged by the standards which its form challenges us to apply, as satisfactory. The "Bibliotheca" is based on a number of catalogues whose character was indicated in the *Gazette* for January of this year, and enriches the observations and quotations which gave to those catalogues their peculiar flavour with a large number of reproductions of old title-pages and wood-cuts. There are two main lists, the first containing 5602 items, the second 10,011, followed by nine groups of addenda with 1784 entries, and by an elaborate subject-index.

Our chief complaint is that the main lists have not been combined or even collated. That lists issued at different times both quoted de Morgan's summing up of Sir Richard Phillips—

"He had four valuable qualities: honesty, zeal, ability, and courage. He applied them all to teaching matters about which he knew nothing, and gained himself an uncomfortable life and a ridiculous memory"—

and both repeated the Admiralty's official answer to the offer by Sir Francis Ronalds in 1816 of his electric telegraph—

"Now the French war is over, telegraphs of any kind are wholly needless, and no other than the semaphore now in use will be adopted"—

is no excuse for having printed these and hundreds of other notes twice, and

several notes more than twice, in this one work, and collation would have discovered a number of regrettable lapses from alphabetical order within the individual lists.

The index would have been more useful if the entries under the headings had been arranged alphabetically instead of following the order of the items in the body of the work. Misprints are astonishingly rare; the most persistent that we have remarked is in the word Liège, which occurs comparatively often, because among the comments transcribed faithfully from the original catalogues are many that were dictated by the passions of war-time. Whether these comments deserve preservation is doubtful, but it is certain that all notes which when written referred to current events should have been dated, as should also such a eulogy as Airy's on T. G. Taylor's "Madras Catalogue of Fixed Stars."

The description on pp. 826 and 858 of the man who was called "le cerveau vivant des sciences rationelles," as "the cousin of the President of the French Republic," implies a scale of values that we should not expect Mr. Sotheran to adopt, but it was at least accurate when it was used, whereas on p. 673 the men are called brothers; the wonder is that in a work full of bibliographical and biographical details there are so few slips. To the buyer of books this "Bibliotheca" is indispensable, and we feel churlish in suggesting to the ill-paid teacher that he can afford to do without it when on opening at random for concluding examples of the entertainment it affords, we alight first on mention of that credulous sceptic who would believe anything provided it was not in the Bible, and then on a theory that dram-drinking is an instinctive racial preparation for the fall of earth-temperature with the cooling of the sun, when living organisms are doomed unless alcohol has taken the place now occupied by water in their economy!

A Treatise on the Integral Calculus, with Applications, Examples, and Problems. By J. EDWARDS. Vol. I. Pp. xxii+907. 50s. 1921. (Macmillan.)

By claiming for this unfortunate book a place in a class to which it does not belong, the author compels the use of a standard which the critic regrets to apply. Since the avowed aim is to give the reader "a good working knowledge of the subject, both practical and theoretical, and to place before him . . . abundance of . . . instances of the application of the principles explained," it is impossible to ignore the quality of the analysis, although the space devoted to general theorems is comparatively small.

It is typical of the theoretical work that 'proof' is given on p. 44 that $\Sigma\int u_n dx = \int \Sigma u_n dx$, provided that Σu_n and $\Sigma\int u_n dx$ are both absolutely (!) convergent, and on p. 335 that mere convergence of Σu_n implies convergence of $\Sigma\int u_n dx$ to the limit $\int \Sigma u_n dx$. On this level, differentiation and integration under the sign of integration naturally present no recognisable difficulties; in the bookwork on p. 362 the range is said to be finite, but no excuse is offered when the process is applied on pp. 369-371 to integrals of infinite range. We can but sympathise with the student who is asked in the Examples at the close of this worthless chapter to discover for himself the explanation of some of the paradoxes that result when illegitimate operations are performed.

It is almost fair to say, in spite of these examples, that to Mr. Edwards all convergence is uniform and all double limits commutative; on p. 361 what is essentially a passage from $\lim_{k\to 0}\int \phi(x,k)dx$ to $\int \lim_{k\to 0}\phi(x,k)dx$ is made without comment. And the blindness extends to fallacies far less subtle than infest double-limit problems. Thus on p. 153: "We have just enough equations to obtain the n quantities, provided that these equations are all independent. But as we have established *otherwise* a means of finding these quantities we may *infer the independence of the equations.*" Will it be believed that the damning italics are not ours, but the author's?

To say that Mr. Edwards is a practical man is not to excuse blunders of this kind. It was open to him to confine his attention to the practical side of the

subject, referring students elsewhere for all theoretical discussions. Instead he has admitted numerous invalid 'proofs' of untrue 'theorems,' and his readers must be warned to scrutinise every argument put forward.

We turn with relief to the manipulative work that gives the volume its value. Of integration, regarded as the converse of differentiation, the account is comprehensive, and the table of contents is an adequate guide. Specially welcome are the exposition in detail of Greenhill's method and a chapter explaining the notation of Jacobian elliptic functions. The absence of reference to Hermite's work on the integration of rational functions is easy to understand, but the alternative to Greenhill's method advocated by Bromwich might have been mentioned, if only in examples, and a chapter to put the Weierstrassian function on the same footing as the Jacobian would not have been out of place.

Problems of geometrical origin occupy more than half of the volume, and it is in notable contrast to the author's treatise on the differential calculus that very little of this work is extraneous to the main purpose of the book. Among the subjects treated, in addition to the quadrature and rectification of specific curves, and to the relations between lengths and between areas connected with various associated loci, are the group of theorems of which the best known is Holditch's, certain parts of the theory of roulettes and glissettes, and the measurement of volumes and of areas of curved surfaces by means of curvilinear coordinates. Contiguous geometrical elements are used as a matter of course, and the only hint that the *existence* of lengths and areas is a subject for discussion is an apt reference to De Morgan, but this part of the treatise is a mine of *information* from which grateful teachers will draw for many years.

A few details call for remark. The assumption that the lower limit of an integral is algebraically the smaller is nowhere formally abandoned. We find $\int dx/x$ and $\int dx/\sqrt{(a^2-x^2)}$ given with the common inaccuracy, and we are told of plane curves that "the value of the radius of curvature is, of course, essentially positive," and that therefore ρ is $+ds/d\psi$ or $-ds/d\psi$ according to circumstances. There seems little point in referring the inequality

$$\int_0^{\frac{1}{3}\pi} \sin x\, dx/x > \int_{\frac{1}{3}\pi}^{\pi} \sin x\, dx/x$$

to Math. Trip., Pt. I., 1912, or in allowing the name of reduction formula to cover every integration by parts. Ex. 30 on p. 104 requires the bookwork on p. 226, and on p. 616 Clerk Maxwell is given a title and a hyphen. It is only occasionally that the language is as careless as when the discussion of quadrature by means of polar coordinates concludes: "If there be a loop and the origin be a singular point on the curve at which the tangents make an angle 2α with each other, and if the initial line be an axis of symmetry, the limits for the area of the loop will be 0 and α and double the result." But the spelling "glisette" is indefensible.

A word of praise is due to the printer: the elaborate typesetting is admirably done, the figures are clear, and misprints are surprisingly rare. In spite of all defects, since the treatise will seldom fall into the hands of students young enough to be harmed by the pernicious principles, the most scrupulous teacher can welcome without reservation the opportunity to benefit by the author's unrivalled knowledge of the problems of integration and by his long experience. E. H. N.

Coordinate Geometry (Plane and Solid) for Beginners. By R. C. Fawdry, M.A., B.Sc. Pp. viii, 215. (G. Bell & Sons.)

This text-book is not written for mathematical specialists, but for engineers and army candidates who need only a working knowledge of the principle of elementary analysis and an acquaintance with the more important properties of Conic Sections. The book should be very useful for this purpose and also for pupils who require only a limited knowledge of the subject for such examinations as Intermediate Science; we should not, however, recommend it to beginners who intend to pursue the subject beyond the limits of this book, for the text deals with the representation of geometrical properties by algebraical

equations rather than with algebraical analysis. A fair knowledge of algebra and trigonometry is assumed. Chap. III. is devoted to the beginning of Calculus, so that the reader may use gradients to find the equations of tangents. Chap. V. gives a short sketchy account of determinants, and Chap. X. deals with three dimensional coordinates. The exercises are numerous and contain an unusually large proportion of numerical examples; they are necessarily of a rather restricted character.

A First Course in the Calculus. Part II. By W. P. MILNE, M.A., D.Sc., and G. J. B. WESTCOTT, M.A. Pp. xv, 181-402, xxxix. 5s. 1921. (G. Bell & Sons.)

Calculus for Beginners. By H. SYDNEY JONES, M.A. Pp. ix, 300. 6s. 1921. (Macmillan & Co.)

Elementary Calculus. By WILLIAM F. OSGOOD, Ph.D., LL.D. Pp. ix, 224. 12s. 6d. 1921. (The Macmillan Co.)

These works aim at applying the methods of the calculus to practical problems from the earliest stages: they are intended for the practical scientist rather than for the mathematical student. Criticism from the purely mathematical standpoint is disarmed by the explanatory prefaces. Considering the size of these books a large programme is attempted, all the usual school calculus, differential and integral, is included in the first two, with a short treatment of differential equations; the third book is confined to differential calculus.

Dr. Milne's book is a continuation of the excellent little text-book which has been in use for some time; the first part contained only such work as could be done by differentiating and integrating powers of x, the second part extends this to work involving trigonometric and logarithmic functions of x. As in Part I. the text is based very largely on graphical work, it may be questioned whether this idea is not carried too far; the underlying principle may easily be lost sight of amidst the crowd of arithmetical calculations involved; to many students graph drawing speedily becomes tedious and irksome. It might perhaps have been well to decrease the heavy arithmetical work as the subject was advanced; the point of many illustrative examples is lost by the introduction of unnecessary numerical work. This, however, is a minor defect; we have used the book for "science" pupils and found the exercises excellent and the bookwork satisfactory.

Some obvious misprints occur (*e.g.* $d \sin x/dx = 1$ on p. 185), and it is unfortunate that the first reference mark in the book is omitted. We should like to see the words "it is plain" and "plainly" deleted throughout; they are superfluous, and frequently appear to beg the question. Dr. Milne's belief in our knowledge of historical dates is touching. We thoroughly recommend this book for use with the right kind of pupil.

Mr. Jones' book proceeds much more rapidly; the trigonometrical functions are introduced on p. 56. A much earlier and more frequent appeal is made to the idea of a "limit"; on the whole we prefer Dr. Milne's more cautious approach. Quite elaborate differentiation occurs quite early in this work, though it may be noted that such exercises frequently lead up to harder integral formulae. A large variety of exercises is provided, and the book should prove useful for the older kind of pupil found in evening classes. It is well printed, but appears on many pages to be a little cramped.

Mr. Osgood confines his work to the differential, and the book is intended for students with a larger range of mathematical knowledge; a discussion of "limits" and the elementary propositions for their sum, product and quotient occurs on p. 25. The subject is developed thenceforward in quite the classic style. In Chap. VII. a useful account of the method of obtaining numerical solutions of equations is given and a short sketch of curve plotting.

The most interesting point which arises from the comparison of these text-books is the widely different treatment given to the differential of a logarithm. Dr. Milne follows Prof. Perry; he finds the gradient of $\log_{10} x$ when $x = 1$ by the use of logarithmic tables. Then by a rather artificial method he finds the ratio of the gradients when the variable is x and X, obtaining

$$(d \log_{10} x)/dx = 0{\cdot}4343/x.$$

He then states that if the base is e, the gradient of $\log_e x$ when $x = 1$ is 1, and so on. The continued use of the base 10 leads, we think, to unnecessary confusion.

Mr. Jones follows the method used in *Calculus Made Easy*, applying the Compound Interest Law. He finds an infinite series in $x(y)$, which is unaltered by differentiation; then using $\Delta y = y\,\Delta x$, he deduces that y is e^x. The process will, we fear, strike the student as rather artificial.

Mr. Osgood, after giving a short sketch of the properties of logarithms, practically follows the method given in Williamson's *Calculus* from the "limit" definition of a differential coefficient. CLEMENT JONES.

CORRESPONDENCE.

2 UPPER CAPE, WARWICK.

To the Editor of the *Mathematical Gazette*.

Sir,

In the October number of the *Gazette* you print a detailed criticism of Howard and Bingham's *School Geometry*. I venture to suggest that the reviewer, perhaps unconsciously, has managed to convey a totally wrong impression of the book. To begin with, the alleged faults of the work are the only points fully discussed; beyond a glowing tribute to the printers, no mention is made of its many obvious merits. But apart from this, the reviewer makes certain statements, quite casually, which seem to imply a great deal that is misleading and damaging.

His second sentence, indeed, is a simple misstatement: "In the opening chapters the alternate angle property of parallel lines is assumed. . . . Would it not be more natural to assume the equality of corresponding angles?" The property assumed in the book *is* the equality of corresponding angles. It is surprising that a criticism so unusually thorough should contain such an error.

"Altogether there are 100 propositions, 13 being problems." The implication is that this part of the work is passed over too lightly. No reference is made to the numerous problems worked up to in the text and set in the exercises.

"It is difficult, however, to find anything of the nature of solid geometry between the first few pages and the last chapter." The authors clearly state in the preface that the bulk of the work in solid geometry is placed together at the end in order that the teacher may use his discretion as to when to introduce it.

These matters are trifling; but they constitute the basis of most of the hostile criticism. The rest is directed at one unimportant slip, a printer's error, and one of the exercises in the early part of the book. It is a pity that the reviewer's passion for accuracy in small details does not extend to his own statements. I have used the *School Geometry* ever since it was first published, and know of no other that compares with it for simplicity, completeness, and wealth of examples; yet any person would be entirely misled by this review of it.

The final jeer is rather nauseating. The "astronomical" exercise, for which the reviewer reserves the full force of his wit, is quite near enough to the truth to illustrate vividly for small people the idea of rotation; and not important enough to affect the future study of astronomy by the boys of Warwick School; but surely (and this is a very serious question) that paragraph of somewhat lumbering humour is in very doubtful taste?—I am, Sir,

Yours sincerely, J. M. HENDRIE.

Dear Sir,

Mr. Hendrie complains of a misstatement in my review of Messrs. Howard and Bingham's *School Geometry*. But he gives a somewhat different colour to the statement complained of by omitting certain words. I wrote: "In the opening chapters the alternate-angle property of parallel lines is assumed and quoted as 'our parallel axiom.'" The authors, in their preface, write: "The alternate-angle property of parallel lines is assumed as axiomatic in the second chapter"; and on page 131, "It will be remembered that we have so far assumed as axiomatic that, when two parallel lines are crossed by a transversal, the alternate angles so formed are equal." The actual treatment in the text is, however, as follows: On page 5 *parallel lines* are defined as "lines in the same plane which never meet." On page 20 the equality-of-corresponding-angles property of parallel lines is assumed, and on the next page the alternate-angle property is deduced. It will be seen that the emphasis is on the alternate-angle property. I am glad to see that the answer to my question "Would it not be more natural to assume the equality of corresponding angles?" is in the affirmative. This explanation will, I trust, remove any possible misconception, and I am sorry if my brief statement has conveyed to any reader a wrong impression of the book.

Concerning the other points in Mr. Hendrie's letter I have little to say. He reads a meaning into my statement, "Altogether there are 100 propositions, 13 being problems," which was certainly not intended, and, passing over the serious criticisms, describes my lighter comments as "nauseating" and "in very doubtful taste." If they are so, they will meet with the condemnation which they deserve.

The preface led me to expect that the recommendations of the Board of Education would be appreciated and acted upon, but a text-book such as this, in my opinion, lends support to the complaints that reform in the teaching of geometry is a failure and must be abandoned.

Mr. Hendrie has written a review very different from mine. He is well able to do so, for he "read many of the proof sheets" (*vide* Preface), and has used the book "ever since it was *first* published." He approves of the contemplated common-sense sun-dial as suitable "for small people."

Is it better, Mr. Editor, for a reviewer to damn a book with faint praise, or to praise it with feigned damns? I do not know, and it worries me. One thing, however, I *do* know; and it is that the reform of geometrical teaching is of far greater importance than the success of any text-book, and my opinion of this book is set out, I think, in due proportion, and without bias, in the review which you printed.—I am, Dear Sir, Yours sincerely,

THE REVIEWER.

31*st October*, 1921.

P.S.—On a certain day the sun crosses the meridian of Warwick School at 12 o'clock noon. Assuming that the time of year is that of an equinox, and that the sun does not sensibly depart from the equator during that afternoon, show that at x o'clock p.m. the reading of the "contemplated common-sense sun-dial" will be y o'dial, where $\tan 15y^\circ = \operatorname{cosec} \phi \tan 15x^\circ$, ϕ being the latitude of Warwick School. Hence obtain the following approximate table:

x	0	1	2	3	4	5	6
y	0	1·24	2·41	3·44	4·36	5·20	6

Also draw the corresponding graph.

Show that

(i) the dial is at no time during that afternoon more than half an hour fast,

(ii) the sun is South-west very shortly after half-past two,

(iii) at 3 o'clock the sun is about $38\frac{1}{2}°$ South of West,
(iv) at 1 o'clock the sun is about $18\frac{1}{2}°$ West of South,
(v) at 4 o'clock the sun is about $24\frac{1}{2}°$ South of West.

P.P.S.—It has come to my knowledge that the Khartum Academy for Small People also contemplates the erection of a "common-sense sun-dial." In the above problem substitute the words "Khartum Academy for Small People" for "Warwick School," and obtain the following approximate table:

x	0	1	2	3	4	5	6
y	0	2·99	4·34	5·00	5·41	5·73	6

Draw the graph and show that
(i) the dial is at no time during that afternoon more than $2\frac{1}{2}$ hours fast,
(ii) the sun is South-west about 1 o'clock,
(iii) at 3 o'clock the sun is about 15° South of West,
(iv) at 4 o'clock the sun is about 9° South of West.

THE REVIEWER.

PURCHASE.

Wanted—*Mathematische Annalen*, Vols. 44–69.—Apply, stating price, carriage paid, to J. D. Williams, M.A., The General Library, University College of Wales, Aberystwith.

ERRATUM.

P. 331, l. 32, *for* "place" *read* "pace."

THE LIBRARY.

THE Library is at 9 Brunswick Square, W.C., the premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

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The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. HOBSON. Vol. I. Second Edition. Pp. xvi+671. 45s. net. (Cam. Univ. Press.)

Bibliotheca Chemico-Mathematica. Catalogue of works in many tongues on Exact and Applied Science. With a Subject Index. Compiled and Annotated by H.Z. and H.C.S. Vols I. and II. Pp. 965. With 127 Plates, containing 247 Portraits and Facsimiles. 3 gs. net. 1921. (Sotheran, 140 Strand.)

Physical and Chemical Constants, and some Mathematical Functions. By G. W. C. KAYE and T. H. LABY. Pp. 161. Fourth Edition. 14s. net. 1920. (Longmans, Green.)

Textile Mathematics. Part II. By T. WOODHOUSE and A. BRAND. Pp. 123. 2s. 6d. 1921. (Blackie.)

The Group-Theory Element of the History of Mathematics. By G. A. MILLER. Pp. 75-82. [Reprint *Sci. Monthly*, Jan. 1921.]

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The Essentials of Mental Measurement. By W. BROWN and G. H. THOMSON. Pp. x+216. 1920. 21s. net. (Cam. Univ. Press.)

Three Lectures on Fermat's Last Theorem. By E. J. MORDELL. Pp. 31. 4s. net. 1920. (Cam. Univ. Press.)

Graphs showing relations between the Object and Image Distances of Thin Lenses and Spherical Mirrors. (With full explanations.) By MOHD. A. R. KHAN. Pp. 8. (From the author, Prof. Khan, B.Sc., Nizam College, Hyderabad, Deccan.) 1921.

The Dynamics of the Airplane. By K. P. WILLIAMS. Pp. viii+138. 13s. 6d. net. 1921. (Wiley: Chapman & Hall.)

Les Contributions des différents peuples au développement des Mathématiques. Pp. 169-194. By GINO LORIA. Reprinted from *Scientia*, April, 1921.

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Elementary Algebra. Part II. By C. V. DURELL and R. M. WRIGHT. Pp. xxiii+253-551+tables+xlvii-lxxxv. Introduction and Answers, 5s. 6d.; without Introduction and with Selected Answers, 4s. 6d. 1921. (Bell & Sons.)

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Perspective. The Old and New Method. By A. S. PERCIVAL. Pp. 42. 4s. 6d. net. 1921. (Longmans, Green.)

Examples in Differential and Integral Calculus. With Answers. By C. S. JACKSON. Pp. 142. 10s. 6d. net. 1921. (Longmans, Green.)

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On Amicable Numbers and their Generalisations. Pp. 195-200. T. E. MASON. *An Elementary Treatment of Fourier's Series.* Pp. 200-203. G. D. BIRKHOFF. *A Cubic Space connected with the Tetrahedron.* Pp. 203-206. F. D. MURNAGHAN. *Among my Autographs* [Lalande, Louis Monge]. Pp. 207-209. E. D. SMITH. *On the Geometrical Representation of Indeterminate Forms.* Pp. 211-217. W. A. HURWITZ.

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$$\frac{d^2\phi(x)}{dx^2} = f[x, \phi(x), \phi'(x)], \quad \int_0^x g(x, y, \phi(y), \phi'(y))\,dy.$$

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